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Kelvin–Helmholtz instability of finite amplitude

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Non-linear Kelvin–Helmholtz instability, of two parallel horizontal streams of inviscid incompressible fluids under the action of gravity, is studied theoretically. The lower stream is denser and there is surface tension between the streams. Some progressing waves of finite amplitude are found as the development of a slightly unstable wave of infinitesimal amplitude. In particular, the non-linear elevation of the interface between the fluids is calculated. The finite amplitude of the waves does not equilibrate to a constant after a long time, but varies periodically with time. In practice, slight dissipation should lead to equilibration at an amplitude close to a value given by the present theory.

1. Introduction

Recent experiments by Thorpe (1968, 1969) have made it possible to confirm Kelvin's classical linear theory of instability of shear flow between two horizontal parallel streams of fluids (cf. Chandrasekhar 1961, § 101). These experiments carry on beyond the linear development of instability, so the time seems ripe to develop the non-linear theory.

The theoretical problem, both mathematically tractable and a fair model of the experiments, is the instability of the basic flow of inviscid incompressible fluids with velocity, density and pressure

$$\mathbf{U} = \begin{cases} U_2\mathbf{i} \\ U_1\mathbf{i} \end{cases}, \quad \rho = \begin{cases} \rho_2 \\ \rho_1 \end{cases}, \quad P = \begin{cases} p_0 - g\rho_2 z & (z > 0), \\ p_0 - g\rho_1 z & (z < 0), \end{cases} \tag{1}$$

respectively, where z is the height and g the acceleration due to gravity. It is assumed that there is an irrotational perturbation of this flow on each side of the interface with elevation $z = \zeta(x, y, t)$, so that the velocity

$$\mathbf{u} = \mathbf{U} + \nabla\phi_j \quad (j = 1 \text{ for } z < \zeta, j = 2 \text{ for } z > \zeta). \tag{2}$$

Then it can be shown (cf. Chandrasekhar 1961, chapter XI) that the non-linear instability is governed by the following system:

$$\nabla^2\phi_j = 0 \quad (j = 1, 2 \text{ for } z \lessgtr \zeta); \tag{3}$$

$$\nabla\phi_j \to 0 \quad \text{as} \quad z \to \mp\infty; \tag{4}$$

$$\frac{\partial\phi_j}{\partial z} - \left(\frac{\partial\zeta}{\partial t} + U_j\frac{\partial\zeta}{\partial x}\right) = \frac{\partial\phi_j}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi_j}{\partial y}\frac{\partial\zeta}{\partial y} \quad (z = \zeta, j = 1, 2); \tag{5}$$

$$\rho_1\left\{\frac{\partial\phi_1}{\partial t}+U_1\frac{\partial\phi_1}{\partial x}+g\zeta\right\}-\rho_2\left\{\frac{\partial\phi_2}{\partial t}+U_2\frac{\partial\phi_2}{\partial x}+g\zeta\right\}-\gamma\nabla^2\zeta$$
$$= \tfrac{1}{2}\{\rho_2(\nabla\phi_2)^2-\rho_1(\nabla\phi_1)^2\}+\gamma\{-\nabla^2\zeta+[\nabla^2\zeta\{1+(\nabla\zeta)^2\}-\tfrac{1}{2}\nabla\zeta.\nabla(\nabla\zeta)^2]$$
$$\times[1+(\nabla\zeta)^2]^{-\frac{3}{2}}\}\quad (z=\zeta); \quad (6)$$

where γ is the interfacial surface tension.

Next we define the operator

$$Q \equiv \frac{\zeta}{1!}\frac{\partial}{\partial z}+\frac{\zeta^2}{2!}\frac{\partial^2}{\partial z^2}+\frac{\zeta^3}{3!}\frac{\partial^3}{\partial z^3}+\dots, \quad (7)$$

and reduce the interfacial conditions at $z=\zeta$ to those at $z=0$ by Maclaurin's theorem, getting

$$\nabla^2\phi_j = 0 \quad (j=1,2 \text{ for } z\lessgtr 0); \quad (8)$$

$$\nabla\phi_j \to 0 \quad \text{as} \quad z\to\mp\infty; \quad (9)$$

$$-\frac{\partial\phi_1}{\partial z}+\frac{\partial\zeta}{\partial t}+U_1\frac{\partial\zeta}{\partial x} = Q\frac{\partial\phi_1}{\partial z}-\frac{\partial\zeta}{\partial x}(1+Q)\frac{\partial\phi_1}{\partial x}-\frac{\partial\zeta}{\partial y}(1+Q)\frac{\partial\phi_1}{\partial y}\quad (z=0); \quad (10)$$

$$\frac{\partial\phi_2}{\partial z}-\frac{\partial\zeta}{\partial t}-U_2\frac{\partial\zeta}{\partial x} = -Q\frac{\partial\phi_2}{\partial z}+\frac{\partial\zeta}{\partial x}(1+Q)\frac{\partial\phi_2}{\partial x}+\frac{\partial\zeta}{\partial y}(1+Q)\frac{\partial\phi_2}{\partial y}\quad (z=0); \quad (11)$$

$$\rho_1\left(\frac{\partial\phi_1}{\partial t}+U_1\frac{\partial\phi_1}{\partial x}\right)-\rho_2\left(\frac{\partial\phi_2}{\partial t}+U_2\frac{\partial\phi_2}{\partial x}\right)+g(\rho_1-\rho_2)\zeta-\gamma\nabla^2\zeta$$
$$= \gamma\{[\nabla^2\zeta\{1+(\nabla\zeta)^2\}-\tfrac{1}{2}\nabla\zeta.\nabla(\nabla\zeta)^2][1+(\nabla\zeta)^2]^{-\frac{3}{2}}-\nabla^2\zeta\}$$
$$-\rho_1\left\{Q\left(\frac{\partial\phi_1}{\partial t}+U_1\frac{\partial\phi_1}{\partial x}\right)+\tfrac{1}{2}(1+Q)(\nabla\phi_1)^2\right\}+\rho_2\left\{Q\left(\frac{\partial\phi_2}{\partial t}+U_2\frac{\partial\phi_2}{\partial x}\right)\right.$$
$$\left.+\tfrac{1}{2}(1+Q)(\nabla\phi_2)^2\right\}\quad (z=0). \quad (12)$$

The linearized terms have been put on the left-hand side of equations (8)–(12). When the non-linear terms are neglected, equations (10)–(12) can be written as

$$L\boldsymbol{\phi} = 0 \quad (z=0), \quad (13)$$

where the linear operator and column vector are defined as

$$L \equiv \begin{pmatrix} -\dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial t}+U_1\dfrac{\partial}{\partial x} \\ 0 & \dfrac{\partial}{\partial z} & -\left(\dfrac{\partial}{\partial t}+U_2\dfrac{\partial}{\partial x}\right) \\ \rho_1\left(\dfrac{\partial}{\partial t}+U_1\dfrac{\partial}{\partial x}\right) & -\rho_2\left(\dfrac{\partial}{\partial t}+U_2\dfrac{\partial}{\partial x}\right) & g(\rho_1-\rho_2)-\gamma\nabla^2 \end{pmatrix}, \quad \boldsymbol{\phi} \equiv \begin{pmatrix}\phi_1\\ \phi_2\\ \zeta\end{pmatrix}. \quad (14)$$

The general solution of this linearized problem (8), (9), (13) for a typical normal mode of wave-numbers α in the x and β in the y direction is the real part of

$$\boldsymbol{\phi} = \begin{pmatrix} \tilde{\alpha}^{-1}(s+i\alpha U_1)\exp\{i(\alpha x+\beta y)+\tilde{\alpha}z\} \\ -\tilde{\alpha}^{-1}(s+i\alpha U_2)\exp\{i(\alpha x+\beta y)-\tilde{\alpha}z\} \\ \exp\{i(\alpha x+\beta y)\} \end{pmatrix} A(t), \quad (15)$$

where $\tilde{\alpha} \equiv +(\alpha^2+\beta^2)^{\frac{1}{2}}$, A varies with time like e^{st}, and

$$\rho_1(s+i\alpha U_1)^2/\tilde{\alpha}+\rho_2(s+i\alpha U_2)^2/\tilde{\alpha}+g(\rho_1-\rho_2)+\tilde{\alpha}^2\gamma = 0,$$

i.e.
$$s = -\frac{i\alpha(\rho_1 U_1+\rho_2 U_2)}{\rho_1+\rho_2} \pm \left\{\frac{\alpha^2\rho_1\rho_2(U_1-U_2)^2}{(\rho_1+\rho_2)^2} - \frac{\tilde{\alpha}g(\rho_1-\rho_2)}{\rho_1+\rho_2} - \frac{\tilde{\alpha}^3\gamma}{\rho_1+\rho_2}\right\}^{\frac{1}{2}}. \tag{16}$$

It can be seen, as a case of Squire's theorem, that if $U_2 \neq U_1$ the unstable component of given total wave-number $\tilde{\alpha}$ grows most rapidly when it is parallel to the basic flow ($\beta = 0$). Therefore it might be anticipated that non-linear waves are observed to be two-dimensional, and indeed this is essentially correct. Accordingly, we shall assume two-dimensional waves in the vertical plane of (x, z) henceforth.

2. Waves stabilized by buoyancy and surface tension

If surface tension and gravity are sufficiently strong, the flow may be stabilized when $\rho_1 > \rho_2$, with pure imaginary roots s given by (16) for all α. Then it is natural to choose dimensionless variables by use of the length scale

$$l \equiv \{\gamma/g(\rho_1-\rho_2)\}^{\frac{1}{2}}$$

and velocity scale $V \equiv \{g\gamma(\rho_1-\rho_2)/(\rho_1+\rho_2)^2\}^{\frac{1}{4}}$. For convenience, also define $\epsilon \equiv (\rho_1-\rho_2)/(\rho_1+\rho_2)$ and make a Galilean transformation so that $U_2 = -U_1$. To put the two-dimensional form of the system (8)–(12) in dimensionless variables in the usual way, we now replace U_1 by $U_1/V \equiv W$ say, U_2 by $-W$, $\mathbf{u}$ by $\mathbf{u}/V$, x by x/l, t by Vt/l etc. Thus equations (10)–(12) become

$$\boldsymbol{L\phi} = \boldsymbol{N\phi} \quad (z = 0); \tag{17}$$

where

$$L \equiv \begin{pmatrix} -\dfrac{\partial}{\partial z} & 0 & \left(\dfrac{\partial}{\partial t}+W\dfrac{\partial}{\partial x}\right) \\ 0 & \dfrac{\partial}{\partial z} & -\left(\dfrac{\partial}{\partial t}-W\dfrac{\partial}{\partial x}\right) \\ (1+\epsilon)\left(\dfrac{\partial}{\partial t}+W\dfrac{\partial}{\partial x}\right) & -(1-\epsilon)\left(\dfrac{\partial}{\partial t}-W\dfrac{\partial}{\partial x}\right) & 2\left(1-\dfrac{\partial^2}{\partial x^2}\right) \end{pmatrix}, \tag{18}$$

$$(\boldsymbol{N\phi})_1 \equiv Q\frac{\partial\phi_1}{\partial z} - \frac{\partial\zeta}{\partial x}(1+Q)\frac{\partial\phi_1}{\partial x}, \tag{19a}$$

$$(\boldsymbol{N\phi})_2 \equiv -Q\frac{\partial\phi_2}{\partial z} + \frac{\partial\zeta}{\partial x}(1+Q)\frac{\partial\phi_2}{\partial x}, \tag{19b}$$

$$(\boldsymbol{N\phi})_3 \equiv -2\frac{\partial^2\zeta}{\partial x^2}\left\{\frac{3}{2}\left(\frac{\partial\zeta}{\partial x}\right)^2 - \frac{15}{8}\left(\frac{\partial\zeta}{\partial x}\right)^4 + \ldots\right\} - (1+\epsilon)\left\{Q\left(\frac{\partial\phi_1}{\partial t}+W\frac{\partial\phi_1}{\partial x}\right)\right.$$
$$\left. + \tfrac{1}{2}(1+Q)(\nabla\phi_1)^2\right\} + (1-\epsilon)\left\{Q\left(\frac{\partial\phi_2}{\partial t}-W\frac{\partial\phi_2}{\partial x}\right)+\tfrac{1}{2}(1+Q)(\nabla\phi_2)^2\right\}. \tag{19c}$$

The linear stability relation (16) has dimensionless form

$$s = -i\alpha\epsilon W \pm \alpha\{(1-\epsilon^2)W^2 - (1+\alpha^2)/\alpha\}^{\frac{1}{2}}. \tag{20}$$

Therefore the wave is unstable if and only if

$$W > W_c(\alpha) \equiv \{(1+\alpha^2)/\alpha(1-\epsilon^2)\}^{\frac{1}{2}}. \quad (21)$$

Therefore the flow is unstable if and only if

$$W > \min_{0<\alpha<\infty} W_c = \{2/(1-\epsilon^2)\}^{\frac{1}{2}}. \quad (22)$$

As $\epsilon \to 0$ for fixed l, V, the case of interfacial waves between streams of nearly equal density is approached. This is a fair approximation to the experimental conditions of Thorpe (1969) for which ϵ was quite small but W of order one. It also simplifies the theory somewhat; in particular, it gives symmetry between the upper and lower streams such that we may anticipate the existence of non-linear, as well as linear, waves moving with the mean velocity of the basic flow, namely with zero velocity. So we shall put $\epsilon = 0$ to get

$$L_0\boldsymbol{\phi} = N_0\boldsymbol{\phi} \quad (z = 0) \quad (23)$$

from equations (17), where

$$L_0 \equiv \begin{pmatrix} -\dfrac{\partial}{\partial z} & 0 & \left(\dfrac{\partial}{\partial t} + W\dfrac{\partial}{\partial x}\right) \\ 0 & \dfrac{\partial}{\partial z} & -\left(\dfrac{\partial}{\partial t} - W\dfrac{\partial}{\partial x}\right) \\ \left(\dfrac{\partial}{\partial t} + W\dfrac{\partial}{\partial x}\right) & -\left(\dfrac{\partial}{\partial t} - W\dfrac{\partial}{\partial x}\right) & 2\left(1 - \dfrac{\partial^2}{\partial x^2}\right) \end{pmatrix} \quad (24)$$

and N_0 is similarly N with $\epsilon = 0$. Then, by linear theory, a wave grows at the relative rate $s = \alpha(W^2 - W_c^2)^{\frac{1}{2}}$ and is unstable if $W > W_c(\alpha) \equiv \{(1+\alpha^2)/\alpha\}^{\frac{1}{2}}$. The minimum of $W_c(\alpha)$ is $2^{\frac{1}{2}}$, occurring for $\alpha = 1$. Therefore, when W slowly increases from zero, we anticipate that instability will arise as soon as W exceeds $2^{\frac{1}{2}}$, at first growing exponentially as a wave of length 2π, later equilibrating as some non-linear standing wave motion of the same length.

This non-linear wave can be found by the method of normal mode cascade, due to Stuart (1960) and Watson (1960). For small $(W - W_c) > 0$ the linear wave grows exponentially with time at a slow rate, and we seek the subsequent development of the wave with time when non-linearity is important. This is found by perturbing the steady linear solution in the limit as $W \to W_c + 0$. So it is convenient to rearrange equations (23) in the form

$$D_0\boldsymbol{\phi} = M_0\boldsymbol{\phi} \quad (z = 0), \quad (25)$$

with all the small linear terms as well as the non-linear terms in the operator M_0. Thus we define

$$D_0 \equiv \begin{pmatrix} -\dfrac{\partial}{\partial z} & 0 & W_c\dfrac{\partial}{\partial x} \\ 0 & \dfrac{\partial}{\partial z} & W_c\dfrac{\partial}{\partial x} \\ W_c\dfrac{\partial}{\partial x} & W_c\dfrac{\partial}{\partial x} & 2\left(1 - \dfrac{\partial^2}{\partial x^2}\right) \end{pmatrix}, \quad (26)$$

$$(M_0\boldsymbol{\phi})_1 \equiv -\frac{\partial\zeta}{\partial t}-(W-W_c)\frac{\partial\zeta}{\partial x}+Q\frac{\partial\phi_1}{\partial z}-\frac{\partial\zeta}{\partial x}(1+Q)\frac{\partial\phi_1}{\partial x}, \tag{27a}$$

$$(M_0\boldsymbol{\phi})_2 \equiv +\frac{\partial\zeta}{\partial t}-(W-W_c)\frac{\partial\zeta}{\partial x}-Q\frac{\partial\phi_2}{\partial z}+\frac{\partial\zeta}{\partial x}(1+Q)\frac{\partial\phi_2}{\partial x}, \tag{27b}$$

$$\begin{aligned}(M_0\boldsymbol{\phi})_3 \equiv{}& \frac{\partial(-\phi_1+\phi_2)}{\partial t}-(W-W_c)\frac{\partial(\phi_1+\phi_2)}{\partial x}-\frac{\partial^2\zeta}{\partial x^2}\left\{3\left(\frac{\partial\zeta}{\partial x}\right)^2-\frac{15}{4}\left(\frac{\partial\zeta}{\partial x}\right)^4+\ldots\right\}\\ &-\left\{Q\left(\frac{\partial\phi_1}{\partial t}+W\frac{\partial\phi_1}{\partial x}\right)+\tfrac{1}{2}(1+Q)(\nabla\phi_1)^2\right\}\\ &+\left\{Q\left(\frac{\partial\phi_2}{\partial t}-W\frac{\partial\phi_2}{\partial x}\right)+\tfrac{1}{2}(1+Q)(\nabla\phi_2)^2\right\}.\end{aligned} \tag{27c}$$

In the first approximation as $W\to W_c+0$, (25) gives

$$D_0\boldsymbol{\phi} = 0 \quad (z=0). \tag{28}$$

By a suitable translation of the x axis, the general real linear solution of (8), (9), (28) can be written as

$$\boldsymbol{\phi} = \boldsymbol{\phi}' \equiv A(t)\begin{pmatrix} W_c\cos\alpha x\, e^{\alpha z}\\ W_c\cos\alpha x\, e^{-\alpha z}\\ \sin\alpha x\end{pmatrix}. \tag{29}$$

For small $(W-W_c)>0$, the small amplitude A will grow exponentially until non-linearity becomes important and modifies the growth. At the same time the form $\boldsymbol{\phi}$ of the disturbance will be modified. To find these modifications, suppose

$$\boldsymbol{\phi} = \boldsymbol{\phi}'+\boldsymbol{\phi}''+\boldsymbol{\phi}'''+\ldots, \tag{30}$$

where the nth term is $O(A^n)$, for fixed small $(W-W_c)>0$. Now we can iterate the solution, finding an equation of the form

$$D_0\boldsymbol{\phi}'' = (M_0\boldsymbol{\phi})'' \quad (z=0). \tag{31}$$

On evaluation this gives

$$(D\boldsymbol{\phi}'')_1 = -\dot{A}\sin\alpha x-\alpha(W-W_c)A\cos\alpha x+\alpha^2W_cA^2\sin 2\alpha x, \tag{32a}$$

$$(D\boldsymbol{\phi}'')_2 = \dot{A}\sin\alpha x-\alpha(W-W_c)A\cos\alpha x-\alpha^2W_cA^2\sin 2\alpha x, \tag{32b}$$

$$(D\boldsymbol{\phi}'')_3 = 2\alpha(W-W_c)\,W_cA\sin\alpha x \quad \text{at} \quad z=0. \tag{32c}$$

The inhomogeneous linear boundary-value problem (8), (9), (32) for $\boldsymbol{\phi}''$ can be solved conveniently with the aid of a suitable scalar product and an adjoint operator D_0^* of D_0. So for any pair of real vectors

$$\boldsymbol{\phi} = \begin{pmatrix}\phi_1\\ \phi_2\\ \zeta_\phi\end{pmatrix}, \quad \boldsymbol{\psi} = \begin{pmatrix}\psi_1\\ \psi_2\\ \zeta_\psi\end{pmatrix},$$

each representing a flow with period $2\pi/\alpha$ in x, define their scalar product

$$(\boldsymbol{\phi},\boldsymbol{\psi}) \equiv \frac{\alpha}{2\pi}\int_0^{2\pi/\alpha}[\phi_1\psi_1+\phi_2\psi_2+\zeta_\phi\zeta_\psi]_{z=0}\,dx. \tag{33}$$

By Fourier analysis, the scalar product can be seen to be the sum of the scalar products of pairs of components varying sinusoidally with $p\alpha x$ ($p = 0, 1, 2, \ldots$). But if $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ vary sinusoidally with $p\alpha x$, then $\phi_1, \psi_1 \propto e^{p\alpha x}$ and $\phi_2, \psi_2 \propto e^{-p\alpha x}$. Therefore, on integration by parts,

$$(\boldsymbol{\phi}, D_0\boldsymbol{\psi}) = (D_0^*\boldsymbol{\phi}, \boldsymbol{\psi}), \tag{34}$$

where the adjoint of D_0 is

$$D_0^* \equiv \begin{pmatrix} -\dfrac{\partial}{\partial z} & 0 & -W_c\dfrac{\partial}{\partial x} \\ 0 & \dfrac{\partial}{\partial z} & -W_c\dfrac{\partial}{\partial x} \\ -W_c\dfrac{\partial}{\partial x} & -W_c\dfrac{\partial}{\partial x} & 2\left(1-\dfrac{\partial^2}{\partial x^2}\right) \end{pmatrix}. \tag{35}$$

It follows that (34) holds for any pair $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ with period $2\pi/\alpha$ in x.

We need one of the solutions $\boldsymbol{\phi}^*$ of (8), (9) and the adjoint equations

$$D_0^*\boldsymbol{\phi}^* = 0 \quad (z = 0).$$

This merely needs replacement of W_c by $-W_c$ in $\boldsymbol{\phi}'$, just as D_0^* is D_0 with $-W_c$ for W_c. Therefore we may take

$$\boldsymbol{\phi}^* = \begin{pmatrix} -W_c\cos\alpha x\, e^{\alpha z} \\ -W_c\cos\alpha x\, e^{-\alpha z} \\ \sin\alpha x \end{pmatrix}. \tag{36}$$

We are now ready to take the scalar product of equations (32) with $\boldsymbol{\phi}^*$. On use of the adjointness (34) and of $D_0^*\boldsymbol{\phi}^* = 0$ $(z = 0)$, this gives $\alpha(W-W_c)A = 0$. This means that $\alpha(W-W_c)A = o(A^2)$ and must be treated in the next approximation, as will be seen in equation (41). Therefore we have

$$D_0\boldsymbol{\phi}'' = \begin{pmatrix} -\dot{A}\sin\alpha x + \alpha^2 W_c A^2 \sin 2\alpha x \\ \dot{A}\sin\alpha x - \alpha^2 W_c A^2\sin 2\alpha x \\ 0 \end{pmatrix} \quad (z = 0). \tag{37}$$

It is now easy to find a particular integral of (8), (9), (37), namely

$$\boldsymbol{\phi}'' = \begin{pmatrix} \alpha^{-1}\dot{A}\sin\alpha x\, e^{\alpha z} - \tfrac{1}{2}\alpha W_c A^2 \sin 2\alpha x\, e^{2\alpha z} \\ -\alpha^{-1}\dot{A}\sin\alpha x\, e^{-\alpha z} + \tfrac{1}{2}\alpha W_c A^2\sin 2\alpha x\, e^{-2\alpha z} \\ 0 \end{pmatrix}. \tag{38}$$

In fact this gives the general solution $\boldsymbol{\phi}' + \boldsymbol{\phi}''$ to order A^2, because any complementary function of (37) can be absorbed in $\boldsymbol{\phi}'$ by redefinition of A. (An exception occurs when $\alpha = 2^{-\frac{1}{2}}$, in which case an arbitrary multiple of the vectors with rows $-W_c\sin 2\alpha x\, e^{2\alpha z}$, $-W_c \sin 2\alpha x\, e^{-2\alpha z}$, $\cos 2\alpha x$ may be added to $\boldsymbol{\phi}''$. This is because $W_c(\alpha) = W_c(2\alpha)$ when $\alpha = 2^{-\frac{1}{2}}$.)

For the next approximation,

$$D_0\boldsymbol{\phi}''' = (M_0\boldsymbol{\phi})''' \quad (z = 0) \tag{39}$$

$$= \begin{pmatrix} -\alpha(W-W_c)A\cos\alpha x - \alpha A\dot{A}\cos 2\alpha x + \tfrac{1}{8}\alpha^3 W_c A^3(9\cos 3\alpha x - \cos\alpha x) \\ -\alpha(W-W_c)A\cos\alpha x - \alpha A\dot{A}\cos 2\alpha x + \tfrac{1}{8}\alpha^3 W_c A^3(9\cos 3\alpha x - \cos\alpha x) \\ \{2\alpha(W-W_c)W_c A - 2\alpha^{-1}\ddot{A} + \tfrac{1}{4}\alpha^3(3\alpha - 5W_c^2)A^3\}\sin\alpha x \\ \qquad + \tfrac{1}{4}\alpha^3(3\alpha + 7W_c^2)A^3\sin 3\alpha x \end{pmatrix}. \tag{40}$$

The scalar product of $\boldsymbol{\phi}^*$ and equations (40) gives

$$\ddot{A} = 2\alpha^2(W-W_c)W_cA - \tfrac{1}{8}\alpha^3(1+4\alpha^2)A^3, \tag{41}$$

with an error of $O(A^4)$. This equation is similar to that of a particle on a soft spring if $W < W_c$. However, we are interested in the case $W > W_c$.

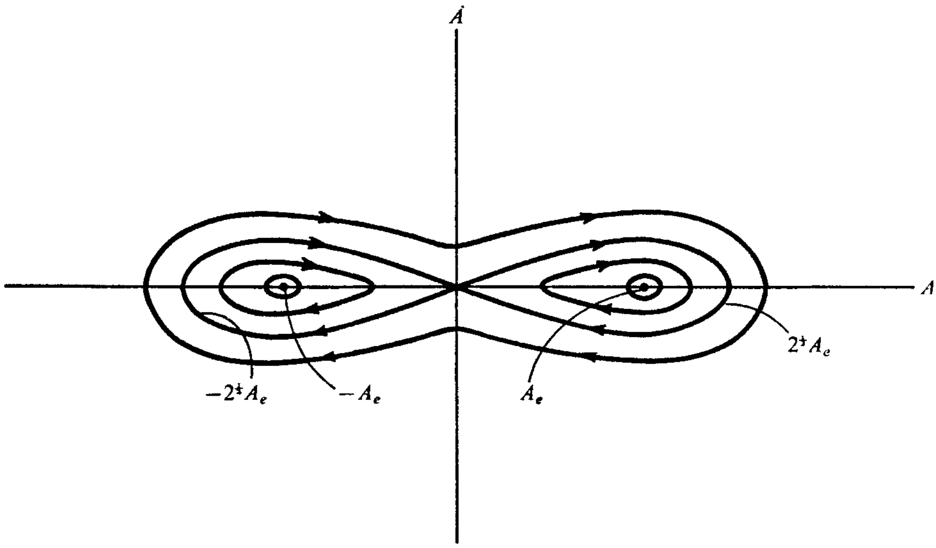

FIGURE 1. The phase plane of A and $\dot{A}$.

Note that $s^2 = 2\alpha^2(W-W_c)W_c + O(W-W_c)^2$, so (41) agrees with the linear theory, whereby $A \propto e^{st}$. Also steady non-linear waves are possible if

$$A^2 = A_e^2 \equiv 16(W-W_c)W_c/\alpha(1+4\alpha^2). \tag{42}$$

For a more detailed study of (41), note that it has 'energy' integral

$$\dot{A}^2 = 2\alpha^2(W-W_c)W_cA^2 - \tfrac{1}{16}\alpha^3(1+4\alpha^2)A^4 + \text{constant}. \tag{43}$$

This can be integrated explicitly in terms of elliptic functions, but the essential features of the solutions are seen more easily in the phase plane of A and $\dot{A}$. The trajectories (43) are drawn in figure 1. It can be seen that the basic flow (1), represented by the origin in the phase plane, is unstable. Any infinitesimal perturbation of the basic state develops into a standing wave, represented by a trajectory passing near the origin in the phase plane. In general this trajectory is an elongated figure of eight, with a narrow waist near the origin and with amplitude $2^{\frac{1}{2}}A_e$ approximately. Exceptionally it is a closed curve like a half of the figure of eight. In either case a standing wave of finite amplitude develops, having wavelength $2\pi/\alpha$ in the x direction. This wave is periodic in time too, because the trajectories in the phase plane are closed, and has maximum amplitude $2^{\frac{1}{2}}A_e$ approximately. The period is long, being longer the smaller the initial disturbance is, and approaching infinity for a trajectory that approaches the origin.

Using relation (41), we find that (40) becomes

$$D_0\boldsymbol{\phi}''' = \begin{pmatrix} -\alpha\{(W-W_c)A+\tfrac{1}{8}\alpha^2W_cA^3\}\cos\alpha x-\alpha A\dot{A}\cos 2\alpha x \\ +\tfrac{9}{8}\alpha^3W_cA^3\cos 3\alpha x \\ -\alpha\{(W-W_c)A+\tfrac{1}{8}\alpha^2W_cA^3\}\cos\alpha x-\alpha A\dot{A}\cos 2\alpha x \\ +\tfrac{9}{8}\alpha^3W_cA^3\cos 3\alpha x \\ -2\alpha W_c\{(W-W_c)A+\tfrac{1}{8}\alpha^2W_cA^3\}\sin\alpha x \\ +\tfrac{1}{4}\alpha^3(3\alpha+7W_c^2)A^3\sin 3\alpha x \end{pmatrix} \quad (z=0). \quad (44)$$

Therefore we may take the solution

$$\boldsymbol{\phi}''' = \begin{pmatrix} \{(W-W_c)A+\tfrac{1}{8}\alpha^2W_cA^3\}\cos\alpha x e^{\alpha z}-\dfrac{1+4\alpha^2}{2(1-2\alpha^2)}A\dot{A}\cos 2\alpha x\, e^{2\alpha z} \\ -\dfrac{\alpha^2W_c(17\alpha^2-4)}{16(3\alpha^2-1)}A^3\cos 3\alpha x\, e^{3\alpha z} \\ \{(W-W_c)A+\tfrac{1}{8}\alpha^2W_cA^3\}\cos\alpha x\, e^{-\alpha z}-\dfrac{1+4\alpha^2}{2(1-2\alpha^2)}A\dot{A}\cos 2\alpha x\, e^{-2\alpha z} \\ -\dfrac{\alpha^2W_c(17\alpha^2-4)}{16(3\alpha^2-1)}A^3\cos 3\alpha x\, e^{-3\alpha z} \\ -\dfrac{\alpha W_c}{1-2\alpha^2}A\dot{A}\sin 2\alpha x-\dfrac{\alpha^2(2-\alpha^2)}{16(3\alpha^2-1)}A^3\sin 3\alpha x \end{pmatrix}. \quad (45)$$

The solution may be continued further in this way, but we shall stop here, now that the leading non-linear behaviour has been found. There are various symmetries in this problem with $\epsilon = 0$; for example, equation (41) in fact has an error of $O(A^5)$, and equations (29), (38) and (45) may be extended to give

$$\zeta = A\sin\alpha x+\frac{\alpha W_c}{2\alpha^2-1}A\dot{A}\sin 2\alpha x-\frac{\alpha^2(2-\alpha^2)}{16(3\alpha^2-1)}A^3\sin 3\alpha x+O(A^5\sin 5\alpha x). \quad (46)$$

This result seems as suitable as any to compare with observations, because the elevation of the interface is one of the least difficult variables to measure. It gives unsteady standing waves. The unsteadiness of the basic flow in the experiments, the viscosity of real fluids, and imperfect cleanliness of their interface may make verification difficult, but these results (41), (46) seem the most susceptible to verification by experiment.

3. Waves stabilized only by buoyancy

Some of Thorpe's (1968) experiments were with smoothly stratified fluid, for which there is a basic flow with smoothly varying velocity and density and without surface tension. However, the discontinuous basic flow (1) that we have used is well known to successfully model many aspects of the linear instability of a smoothly varying flow (cf. Drazin & Howard 1966), particularly the stability characteristics of long waves, so it is desirable to use the basic flow (1) as a model when there is no surface tension.

For this case with $\gamma = 0$, relation (16) gives

$$s = -\frac{i\alpha(\rho_1 U_1 + \rho_2 U_2)}{\rho_1+\rho_2} \pm \left\{\frac{\alpha^2\rho_1\rho_2(U_1-U_2)^2}{(\rho_1+\rho_2)^2} - \frac{\alpha g(\rho_1-\rho_2)}{\rho_1+\rho_2}\right\}^{\frac{1}{2}} \tag{47}$$

for two-dimensional waves, and therefore instability for some (short) waves however strong the buoyancy of the basic flow may be. Here it is more natural to choose a velocity scale $V \equiv \frac{1}{2}(U_1 - U_2)$ and a length scale $l \equiv V^2/g$ for the dimensionless variables. Again, let us suppose that $U_2 = -U_1, = -V$, at first. Then the dimensionless form of the non-linear problem (8)–(12) is

$$\nabla^2\phi_j = 0 \quad (j = 1, 2 \text{ for } z \lessgtr 0); \tag{48}$$

$$\nabla\phi_j \to 0 \quad \text{as} \quad z \to \mp\infty; \tag{49}$$

$$-\frac{\partial\phi_1}{\partial z} + \frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial x} = Q\frac{\partial\phi_1}{\partial z} - \frac{\partial\zeta}{\partial x}(1+Q)\frac{\partial\phi_1}{\partial x} \quad (z = 0); \tag{50}$$

$$\frac{\partial\phi_2}{\partial z} - \frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial x} = -Q\frac{\partial\phi_2}{\partial z} + \frac{\partial\zeta}{\partial x}(1+Q)\frac{\partial\phi_2}{\partial x} \quad (z = 0); \tag{51}$$

$$\begin{aligned}(1+\epsilon)\left(\frac{\partial\phi_1}{\partial t} + \frac{\partial\phi_1}{\partial x}\right) - (1-\epsilon)\left(\frac{\partial\phi_2}{\partial t} - \frac{\partial\phi_2}{\partial x}\right) + 2\epsilon\zeta \\ = -(1+\epsilon)\left\{Q\left(\frac{\partial\phi_1}{\partial t} + \frac{\partial\phi_1}{\partial x}\right) + \tfrac{1}{2}(1+Q)(\nabla\phi_1)^2\right\} \\ +(1-\epsilon)\left\{Q\left(\frac{\partial\phi_2}{\partial t} - \frac{\partial\phi_2}{\partial x}\right) + \tfrac{1}{2}(1+Q)(\nabla\phi_2)^2\right\} \quad (z = 0).\end{aligned} \tag{52}$$

The marginally stable wave in the linear problem for this case is not steady, but a progressive wave with phase velocity $c \equiv is/\alpha = \epsilon$. So we should look not for a standing non-linear wave but rather for one that is steady with respect to an observer moving horizontally with some speed v in the x direction, where $v = \epsilon$ in the linear approximation. To prepare for this we make a Galilean transformation of the system, replacing x by $X \equiv x - vt$ and therefore $\partial/\partial x$ by $\partial/\partial X$ and $\partial/\partial t$ by $\partial/\partial t - v\,\partial/\partial X$. Thus equations (48), (49) are unchanged in form if now $\nabla \equiv (\partial/\partial X, \partial/\partial z)$, but (50), (51), (52) respectively become

$$-\frac{\partial\phi_1}{\partial z} + \frac{\partial\zeta}{\partial t} + (1-v)\frac{\partial\zeta}{\partial X} = Q\frac{\partial\phi_1}{\partial z} - \frac{\partial\zeta}{\partial X}(1+Q)\frac{\partial\phi_1}{\partial X} \quad (z = 0); \tag{53}$$

$$\frac{\partial\phi_2}{\partial z} - \frac{\partial\zeta}{\partial t} + (1+v)\frac{\partial\zeta}{\partial X} = -Q\frac{\partial\phi_2}{\partial z} + \frac{\partial\zeta}{\partial X}(1+Q)\frac{\partial\phi_2}{\partial X} \quad (z = 0); \tag{54}$$

$$\begin{aligned}(1+\epsilon)\left\{\frac{\partial\phi_1}{\partial t} + (1-v)\frac{\partial\phi_1}{\partial X}\right\} - (1-\epsilon)\left\{\frac{\partial\phi_2}{\partial t} - (1+v)\frac{\partial\phi_2}{\partial X}\right\} + 2\epsilon\zeta \\ = -(1+\epsilon)\left[Q\left\{\frac{\partial\phi_1}{\partial t} + (1-v)\frac{\partial\phi_1}{\partial X}\right\} + \tfrac{1}{2}(1+Q)(\nabla\phi_1)^2\right] \\ +(1-\epsilon)\left[Q\left\{\frac{\partial\phi_2}{\partial t} - (1+v)\frac{\partial\phi_2}{\partial X}\right\} + \tfrac{1}{2}(1+Q)(\nabla\phi_2)^2\right] \quad (z = 0).\end{aligned} \tag{55}$$

In this transformed system, the stability relation (47) becomes

$$s = i\alpha(v-\epsilon) \pm \{\alpha^2(1-\epsilon^2)-\alpha\epsilon\}^{\frac{1}{2}}. \tag{56}$$

If
$$v = \epsilon \tag{57}$$

and
$$\epsilon < \epsilon_c(\alpha) \equiv \{(1+4\alpha^2)^{\frac{1}{2}}-1\}/2\alpha, \tag{58}$$

then there are unstable standing waves with relative growth rate

$$s = \alpha\{(\epsilon_c-\epsilon)(1+\epsilon_c\epsilon)/\epsilon_c\}^{\frac{1}{2}} \sim \{(1+\epsilon_c^2)(\epsilon_c-\epsilon)/\epsilon_c\}^{\frac{1}{2}} \quad \text{as} \quad \epsilon \to \epsilon_c - 0. \tag{59}$$

Also note that there are stable *steady* linear waves if $\epsilon \geqslant \epsilon_c$ and

$$v = \epsilon \mp \{\epsilon^2 - 1 + \epsilon/\alpha\}^{\frac{1}{2}}, \tag{60}$$

though only when $\epsilon = \epsilon_c$ is a steady wave on the margin between stable and unstable waves.

To iterate the slightly unstable solution for small A and small $\epsilon_c - \epsilon$ and $v - \epsilon_c$, we first write (53)–(55) with the small terms on the right-hand side as

$$D\boldsymbol{\phi} = M\boldsymbol{\phi} \quad (z=0), \tag{61}$$

where

$$D \equiv \begin{pmatrix} -\dfrac{\partial}{\partial z} & 0 & (1-\epsilon_c)\dfrac{\partial}{\partial X} \\ 0 & \dfrac{\partial}{\partial z} & (1+\epsilon_c)\dfrac{\partial}{\partial X} \\ \dfrac{\partial}{\partial X} & \dfrac{\partial}{\partial X} & 2\alpha \end{pmatrix}, \tag{62}$$

and

$$(M\boldsymbol{\phi})_1 \equiv -\frac{\partial\zeta}{\partial t} + (v-\epsilon_c)\frac{\partial\zeta}{\partial X} + Q\frac{\partial\phi_1}{\partial z} - \frac{\partial\zeta}{\partial X}(1+Q)\frac{\partial\phi_1}{\partial X}, \tag{63a}$$

$$(M\boldsymbol{\phi})_2 \equiv \frac{\partial\zeta}{\partial t} - (v-\epsilon_c)\frac{\partial\zeta}{\partial X} - Q\frac{\partial\phi_2}{\partial z} + \frac{\partial\zeta}{\partial X}(1+Q)\frac{\partial\phi_2}{\partial X}, \tag{63b}$$

$$\begin{aligned}(M\boldsymbol{\phi})_3 \equiv {} & -\frac{1+\epsilon}{1-\epsilon_c^2}(1+Q)\frac{\partial\phi_1}{\partial t} + \frac{1-\epsilon}{1-\epsilon_c^2}(1+Q)\frac{\partial\phi_2}{\partial t} \\ & + \left\{-\frac{\epsilon-\epsilon_c}{1+\epsilon_c} + \frac{v-\epsilon_c}{1-\epsilon_c} + \frac{(\epsilon-\epsilon_c)(v-\epsilon_c)}{1-\epsilon_c^2}\right\}(1+Q)\frac{\partial\phi_1}{\partial X} \\ & + \left\{\frac{\epsilon-\epsilon_c}{1-\epsilon_c} - \frac{v-\epsilon_c}{1+\epsilon_c} + \frac{(\epsilon-\epsilon_c)(v-\epsilon_c)}{1-\epsilon_c^2}\right\}(1+Q)\frac{\partial\phi_2}{\partial X} - \frac{2(\epsilon-\epsilon_c)}{1-\epsilon_c^2}\zeta \\ & - Q\frac{\partial(\phi_1+\phi_2)}{\partial X} + \frac{1}{2(1-\epsilon_c^2)}(1+Q)\{(1-\epsilon)(\nabla\phi_2)^2 - (1+\epsilon)(\nabla\phi_1)^2\}. \end{aligned} \tag{63c}$$

The linear theory gives at once $D\boldsymbol{\phi}' = 0$ $(z=0)$ and thence the first approximation

$$\boldsymbol{\phi}' = A(t)\begin{pmatrix} (1-\epsilon_c)\cos\alpha X\, e^{\alpha z} \\ (1+\epsilon_c)\cos\alpha X\, e^{-\alpha z} \\ \sin\alpha X \end{pmatrix}. \tag{64}$$

Next, retaining terms of order A^2 in (61), we get

$$D\boldsymbol{\phi}'' = (M\boldsymbol{\phi})'' \quad (z = 0) \tag{65}$$

$$= \begin{pmatrix} -\dot{A}\sin\alpha X + \alpha(v-\epsilon_c)A\cos\alpha X + \alpha^2(1-\epsilon_c)A^2\sin 2\alpha X \\ \dot{A}\sin\alpha X - \alpha(v-\epsilon_c)A\cos\alpha X - \alpha^2(1+\epsilon_c)A^2\sin 2\alpha X \\ -2\alpha(1+\epsilon_c^2)\epsilon_c^{-1}(1-\epsilon_c^2)^{-1}(\epsilon-\epsilon_c)A\sin\alpha X + \alpha^2\epsilon_c A^2\cos 2\alpha X \end{pmatrix}. \tag{66}$$

As in §2, we need the adjoint D^* of D. One finds that $(\boldsymbol{\phi}, D\boldsymbol{\psi}) = (D^*\boldsymbol{\phi}, \boldsymbol{\psi})$ for all $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, where now

$$D^* \equiv \begin{pmatrix} -\dfrac{\partial}{\partial z} & 0 & -\dfrac{\partial}{\partial X} \\ 0 & \dfrac{\partial}{\partial z} & -\dfrac{\partial}{\partial X} \\ -(1-\epsilon_c)\dfrac{\partial}{\partial X} & -(1+\epsilon_c)\dfrac{\partial}{\partial X} & 2\alpha \end{pmatrix}. \tag{67}$$

We shall need two solutions of the adjoint system $D^*\boldsymbol{\phi}^* = 0$ $(z = 0)$, and so take the solutions

$$\boldsymbol{\phi}^* \equiv \begin{pmatrix} \cos\alpha X\, e^{\alpha z} \\ \cos\alpha X\, e^{-\alpha z} \\ -\sin\alpha X \end{pmatrix}, \quad \boldsymbol{\psi}^* \equiv \begin{pmatrix} \sin\alpha X\, e^{\alpha z} \\ \sin\alpha X\, e^{-\alpha z} \\ \cos\alpha X \end{pmatrix}. \tag{68}$$

Now the scalar product of $\boldsymbol{\phi}^*$ and equations (66) gives $\epsilon - \epsilon_c = 0$ to the present order of approximation, which means that $\epsilon - \epsilon_c = O(A^2)$ as $A \to 0$. Again, we must delay treatment of $(\epsilon-\epsilon_c)A$ until the next approximation.

This leaves

$$D\boldsymbol{\phi}'' = \begin{pmatrix} -\dot{A}\sin\alpha X + \alpha(v-\epsilon_c)A\cos\alpha X + \alpha^2(1-\epsilon_c)A^2\sin 2\alpha X \\ \dot{A}\sin\alpha X - \alpha(v-\epsilon_c)A\cos\alpha X - \alpha^2(1+\epsilon_c)A^2\sin 2\alpha X \\ \alpha^2\epsilon_c A^2\cos 2\alpha X \end{pmatrix} \quad (z = 0). \tag{69}$$

Therefore one can pick out the particular integral

$$\boldsymbol{\phi}'' = \begin{pmatrix} \{\alpha^{-1}\dot{A}\sin\alpha X - (v-\epsilon_c)A\cos\alpha X\}e^{\alpha z} - \tfrac{1}{2}\epsilon_c A^2\sin 2\alpha X\, e^{2\alpha z} \\ -\{\alpha^{-1}\dot{A}\sin\alpha X - (v-\epsilon_c)A\cos\alpha X\}e^{-\alpha z} + \tfrac{1}{2}\epsilon_c A^2\sin 2\alpha X\, e^{-2\alpha z} \\ \tfrac{1}{2}\alpha\epsilon_c A^2\cos 2\alpha X \end{pmatrix}, \tag{70}$$

which gives the general solution $\boldsymbol{\phi} = \boldsymbol{\phi}' + \boldsymbol{\phi}'' + O(A^3)$ on redefining A by adding to it a term of order A^2.

Taking terms of order A^3 in (61), we find

$$D\boldsymbol{\phi}''' = (M\boldsymbol{\phi})''' \quad (z = 0), \tag{71}$$

and therefore

$$(D\boldsymbol{\phi}''')_1 = -\tfrac{1}{8}\alpha^3(1-\epsilon_c)(1+6\epsilon_c)A^3\cos\alpha X - \alpha(1+\epsilon_c)A\dot{A}\cos 2\alpha X$$
$$- \alpha^2(1+\epsilon_c)(v-\epsilon_c)A^2\sin 2\alpha X + \tfrac{9}{8}\alpha^3(1-\epsilon_c)(1+2\epsilon_c)A^3\cos 3\alpha X, \tag{72a}$$

$$(D\boldsymbol{\phi}''')_2 = -\tfrac{1}{8}\alpha^3(1+\epsilon_c)(1-6\epsilon_c)A^3\cos\alpha X - \alpha(1-\epsilon_c)A\dot{A}\cos 2\alpha X - \alpha^2(1-\epsilon_c)(v-\epsilon_c)A^2\sin 2\alpha X + \tfrac{9}{8}\alpha^3(1+\epsilon_c)(1-2\epsilon_c)A^3\cos 3\alpha X, \quad (72b)$$

$$(D\boldsymbol{\phi}''')_3 = \{-2\alpha(1+\epsilon_c^2)\epsilon_c^{-1}(1-\epsilon_c^2)^{-1}(\epsilon-\epsilon_c)A - 2\alpha^{-1}(1-\epsilon_c^2)^{-1}\ddot{A} - \tfrac{1}{4}\alpha^3(5-2\epsilon_c^2)A^3 + 2\alpha(1-\epsilon_c^2)^{-1}(v-\epsilon_c)^2A\}\sin\alpha X + 4(1-\epsilon_c^2)^{-1}(v-\epsilon_c)\dot{A}\cos\alpha X + \tfrac{1}{4}\alpha^3(7-10\epsilon_c^2)A^3\sin 3\alpha X, \quad (72c)$$

at $z = 0$.

The scalar product of $\boldsymbol{\phi}^*$ with equations (72) gives $(v-\epsilon_c)\dot{A}$ as zero to the present approximation. This means we must put $v = \epsilon_c + O(A^2)$ in order to find a co-ordinate frame with respect to which there is a non-linear standing wave. In other words, to the present order of approximation the phase velocity of the non-linear wave is ϵ_c with respect to the original frame. We may now neglect the term with $(v-\epsilon_c)^2 A$ in equation (72c) for the present approximation. It follows that the scalar product of $\boldsymbol{\phi}^*$ and equations (72) gives, with an error of order A^4,

$$\ddot{A} = \alpha^2(1+\epsilon_c^2)\{(\epsilon_c-\epsilon)\epsilon_c^{-1}A - \tfrac{1}{2}\alpha\epsilon_c A^3\}. \quad (73)$$

This non-linear ordinary differential equation has the same qualitative behaviour as (41) of §2. So again there is a stable steady non-linear wave of amplitude

$$A_e \equiv \{2(\epsilon_c-\epsilon)/\alpha\epsilon_c^2\}^{\frac{1}{2}} \quad (74)$$

as $\epsilon \to \epsilon_c - 0$. However, this steady wave is not the limit of a small unstable disturbance of the basic flow $A = 0$ after a long time. Instead a small disturbance of the basic flow grows into an unsteady standing wave, which has a long period in time and a maximum amplitude of $2^{\frac{1}{2}}A_e$ approximately. These results also give

$$D\boldsymbol{\phi}''' = \begin{pmatrix} -\tfrac{1}{8}\alpha^3(1-\epsilon_c)(1+6\epsilon_c)A^3\cos\alpha X - \alpha(1+\epsilon_c)A\dot{A}\cos 2\alpha X + \tfrac{9}{8}\alpha^3(1-\epsilon_c)(1+2\epsilon_c)A^3\cos 3\alpha X \\ -\tfrac{1}{8}\alpha^3(1+\epsilon_c)(1-6\epsilon_c)A^3\cos\alpha X - \alpha(1-\epsilon_c)A\dot{A}\cos 2\alpha X + \tfrac{9}{8}\alpha^3(1+\epsilon_c)(1-2\epsilon_c)A^3\cos 3\alpha X \\ -\tfrac{1}{4}\alpha^3(1-6\epsilon_c^2)A^3\sin\alpha X + \tfrac{1}{4}\alpha^3(7-10\epsilon_c^2)A^3\sin 3\alpha X \end{pmatrix} \quad (z=0). \quad (75)$$

It follows that

$$\boldsymbol{\phi}''' = \begin{pmatrix} \tfrac{1}{8}\alpha^2(1-\epsilon_c)(1+6\epsilon_c)A^3\cos\alpha X e^{\alpha z} - \tfrac{1}{2}(1-3\epsilon_c)A\dot{A}\cos 2\alpha X e^{2\alpha z} + \tfrac{1}{4}\alpha\epsilon_c(1+2\epsilon_c)A^3\cos 3\alpha X\, e^{3\alpha z} \\ \tfrac{1}{8}\alpha^2(1+\epsilon_c)(1-6\epsilon_c)A^3\cos\alpha X e^{-\alpha z} - \tfrac{1}{2}(1+3\epsilon_c)A\dot{A}\cos 2\alpha X e^{-2\alpha z} - \tfrac{1}{4}\alpha\epsilon_c(1-2\epsilon_c)A^3\cos 3\alpha X\, e^{-3\alpha z} \\ -A\dot{A}\sin 2\alpha X + \tfrac{1}{8}\alpha^2(1-4\epsilon_c^2)A^3\sin 3\alpha X \end{pmatrix}. \quad (76)$$

This, with equations (64), (70), gives

$$\zeta = A\sin\alpha X + \tfrac{1}{2}\alpha\epsilon_c A^2\cos 2\alpha X - A\dot{A}\sin 2\alpha X + \tfrac{1}{8}\alpha^2(1-4\epsilon_c^2)A^3\sin 3\alpha X + O(A^4) \quad \text{as} \quad \epsilon \to \epsilon_c - 0. \quad (77)$$

Here the term $-A\dot{A}\sin 2\alpha X$ introduces some asymmetry of the waves about their crests, much as the similar term in (46); when A^2 is increasing with time, the centre of a half wavelength has lesser x co-ordinate than the crest of the wave. This is in accord with one's intuition that the crest of the wave bends with the stream of the basic flow.

Even if the basic flow (1) were realized in an experiment, the above finite amplitude disturbances would not be observed, because the flow is always unstable to short waves, whatever the value of ϵ. However, the discontinuous basic flow (1) does model the stability characteristics of a shear layer with smoothly varying velocity and density (cf. Drazin & Howard 1966). A smoothly varying shear layer can be completely stabilized by buoyancy when the Richardson number is everywhere greater than a quarter, and so should have marginally unstable finite amplitude waves similar to those we have described above. Thus it is plausible that our results may be used qualitatively but not quantitatively to describe the onset of instability in a shear layer. In particular, they give the non-linear deformation of layers of fluid particles of equal density, and show that instability grows into an unsteady non-linear wave steadily progressing in the direction of the denser fluid.

4. Discussion

On Landau's theoretical grounds (cf. Landau & Lifshitz 1959, §27) the spectrum of disturbances evolves towards turbulence as W increases substantially above W_c. However, Thorpe's (1968) photographs suggest that the non-linear wave itself is not unstable until the vortex sheet has rolled up, by which time ζ is no longer a single-valued function of x. So the representation of ζ as a Fourier series would be unable to describe much of the spectral evolution.

Also the differences of the form of (41), (73) from that of Landau's equation,

$$\frac{d|A|^2}{dt} = 2s|A|^2 - l|A|^4, \tag{78}$$

should be noted. Any solution of Landau's equation equilibrates such that

$$|A| \to A_e \equiv (2s/l)^{\frac{1}{2}} \quad \text{as} \quad t \to \infty \quad \text{if} \quad s, l > 0.$$

However, this is not so for equations (41), (73). This difference is due to the presence of a second, rather than a first, time derivative in each of (41), (73). They are characteristic of the non-linear instability of plane parallel flows with buoyancy, because the equation governing the linear instability of these flows has a second time derivative. For example, the Taylor–Goldstein equation (cf. Drazin & Howard 1966, equation (5.1)) governing the instability of the basic flow $U(z)\mathbf{i}$ of inviscid incompressible fluid with density $\rho_0(z)$ is essentially

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left\{\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi' - \frac{d^2U}{dz^2}\frac{\partial\psi'}{\partial x}\right\} - \frac{g}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial^2\psi'}{\partial x^2} = 0, \tag{79}$$

where ψ' is the stream function of the two-dimensional perturbation.

In practice a little viscosity would dissipate the energy of the disturbance, albeit slowly. This would drastically change the form of the trajectories of the phase plane, figure 1. In particular, the points of equilibrium at $A = \pm A_e$ would become stable spiral points, and each trajectory would approach one of these points as $t \to +\infty$. Thus there is equilibration with wave amplitude A_e for a fluid with infinitesimal viscosity. This is qualitatively modelled by the equation of self-interaction for Bénard convection (Palm 1960, equation (6.17) with $A_{021} = 0$).

It was found in §3 that $v = \epsilon$ to the highest approximation attained. Indeed, one might have put $v = \epsilon$ at first for all $\epsilon - \epsilon_c$ by direct use of the method of Stuart (1960) and Watson (1960). However, the method used above with a Galilean transformation is instructive. In particular, it gives the non-linear development of the stable linear disturbances when $\epsilon > \epsilon_c$. For v can always be chosen so that one root s of (56) is zero when $\epsilon > \epsilon_c$. The non-linear development of such a wave is given by equations (72), there being steady non-linear waves with amplitudes

$$A = \left[\frac{2}{\alpha\epsilon_c}\left\{\frac{\epsilon_c - \epsilon}{\epsilon_c} + \frac{(v-\epsilon_c)^2}{1+\epsilon_c^2}\right\}\right]^{\frac{1}{2}}. \tag{80}$$

One may speculate on the relevance of this work to the generation of waves by wind. Of course the relevance of the model to a turbulent wind is little more than speculation. Under some circumstances the Kelvin–Helmholtz model may be relevant to ocean waves (Miles 1959), in which case non-linear mechanisms deserve consideration.

Finally, we summarise this work, whose physical significance may be obscured by the large amount of algebra and calculus that has been necessary. The idealizations of the theoretical model—inviscid fluid, unbounded and steady horizontal uniform streams, clean interface, infinitesimal rate of growth of instability, two-dimensional disturbance—cannot be met in real experiments. But these idealizations have led to results which we anticipate to be qualitatively applicable to appropriate real flows. The linear solution for a slightly unstable mode has been iterated to account for non-linear self-interaction, and thereby it was shown that the mode did not break down into turbulence directly and rapidly but rather became a periodic non-linear wave. It was suggested that this non-linear wave would become steady if the fluid had viscosity, however small. These results are not expected to apply to the development of a very unstable linear wave, but should be applied to the wave-number of the most unstable linear disturbance in slightly supercritical conditions; thus $\alpha = 1, 0 < W - 2^{\frac{1}{2}} \ll 1$ in the problem of §2.

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